Then Eqs. (11) and (14) may be written more simply as

$$(df/d\epsilon) = (\partial G/\partial u_k)w_k \quad (17)$$

and

$$\frac{d^2f}{d\epsilon^2} = \frac{\partial^2 G}{\partial u_l \partial u_k} w_k w_l + \frac{\partial^2 G}{\partial y_i \partial u_k} w_k h_i + \frac{\partial^2 G}{\partial u_l \partial y_i} h_i w_l + \frac{\partial^2 G}{\partial y_j \partial y_i} h_i h_j \quad (18)$$

We see from Eq. (18) that the condition $d^2f/d\epsilon^2 > 0$, is a condition on a quadratic form which involves both dependent and independent variations. That is, in evaluating $d^2f/d\epsilon^2$, h_i, and w_k are related by the linear Eq. (10). Hence, two options are open before applying Lemma 1. Option 1: Eq. (10) may be used to eliminate the h_i in Eq. (18); Option 2: the following theorem may be used.

Theorem 2

The maximum and minimum values of the form $x^T\mathbf{A}x/||x||^2$ subject to the conditions $\mathbf{Bx} = \mathbf{0}$, are the zero's of $D(e)$, where

$$D(e) = \begin{vmatrix} \mathbf{A} - e\mathbf{I} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{vmatrix} \quad (19)$$

For proof see Hestenes (Ref. 4, p. 33).

Hence, it follows if the roots of $D(e) = 0$ are all positive, the quadratic form $x^T\mathbf{A}x$ is positive definite. Under option 1, the necessary and sufficient conditions for constrained extrema reduce to those given in the recent work of Bryson and Ho (Ref. 6, p. 9) and under option 2 the results of Hancock (Ref. 5, p. 116) are obtained.

By making a few observations, a multiplier rule (akin to those normally stated) may now be given which alludes to both necessary and sufficiency conditions. These observations are the following: 1) The quantity to be extremized must contain at least one state parameter. (Otherwise the constraint equations would have no bearing on the problem, i.e., f could then be extremized independently of the equations of constraint.) Thus, at least some multipliers will be defined by Eq. (15). 2) The conditions given under either option 1 or 2 are identical to those required for the function $G(y_i,u_k)$, subject to g_j, to be a minimum for the proper choice of multipliers.

A multiplier rule

If y_i^p, u_k^p is an ordinary extremal point of $f(y_i,u_k)$ subject to $g_j(y_i,u_k)$ both of class C^2 and if

$$\partial g_i/\partial y_i|y_i^p,u_k^p \neq 0 \quad (20)$$

then there exists multipliers λ_j, such that y_i^p, u_k^p is also an ordinary extremal point of the function $G(y_i,u_k) = f(y_i,u_k) + \lambda_j g_j(y_i,u_k)$, subject to $g_j(y_i,u_k)$.§

It also follows that an alternate and weaker form of this rule which allows G to be thought of as a function λ_j is that there exists multipliers λ_j not all zero, such that y_i^p, u_k^p is a stationary point of the function $G(y_i,u_k,\lambda_j) = f(y_i,u_k) + \lambda_j g_j(y_i,u_k)$ subject to no constraints.

III. Conclusions

It is concluded that Lagrange multipliers do not possess the properties sometimes ascribed with their use. Using them to adjoin constraint equations to the quantity to be extremized to form an augmented function does not imply that optimizing conditions for the augmented function with no constraints are equivalent to those of the original problem. The Lagrange multipliers do not in effect unconstrain the constrained variables in the problem. Even though the necessary and sufficient conditions can be expressed completely without the use of Lagrange multipliers, if so desired, their use does allow for a convenient formalism in terms of a multiplier rule.

§ It could be argued that this result is obviously true from the onset, since for any point y_i, u_k satisfying $g_j = 0$, the value of f is equal to the value of G.

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Use of Rouse's Stability Parameter in Determining the Critical Layer Height of a Laminar Boundary Layer

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Nomenclature

f_w = similarity parameter evaluated at wall conditions
He = freestream total enthalpy
h_w = enthalpy based on wall temperature
j = {1 axisymmetric flow; 0 two-dimensional flow
Me = Mach number at edge of boundary layer
Pr = Prandtl number
R = Rouse's stability parameter
$R_{e,x}$ = Reynolds number based on boundary-layer edge conditions and distance from leading edge or stagnation point
T_0 = total temperature
t_w = h_w/He
u = velocity in x direction
x,y = coordinate axes
y_c = critical height
β = pressure gradient parameter
γ = ratio of specific heats
δ = boundary-layer thickness
ρ = density
μ = dynamic viscosity

In connection with boundary-layer transition studies, several investigators[1–7] have measured the location of the maximum mean square output of hot wire or hot film anemometers in laminar boundary layers. The location of this maximum output has been assumed to represent the critical layer in the boundary layer where transition is most likely to be initiated. The characteristics of this critical layer have been used to "explain" some of the properties of boundary-

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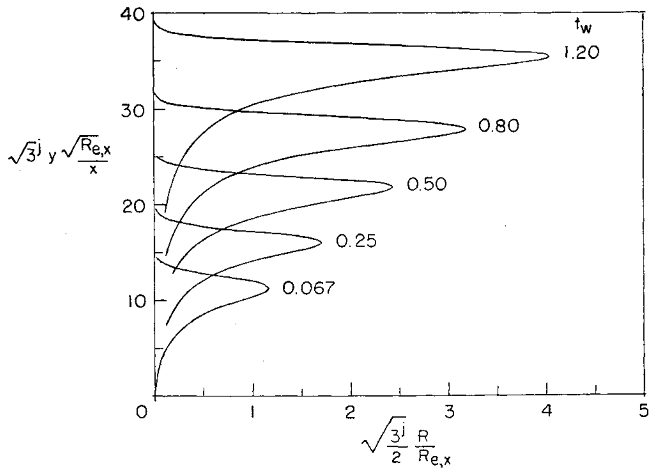

Fig. 1 Variation of Rouse's stability parameter through a laminar boundary layer, $\beta = 0$, $T_0 = 1500°$ R, $Pr = 0.70$, $Me = 8, f_w = 0$, $\gamma = 1.4$.

layer transition. As an example, Nagamatsu[5] has suggested that the observed[8,9] insensitivity of hypersonic boundary-layer transition Reynolds numbers to wall cooling effects may be related to the fact that the critical layer is located near the outer edge of the boundary layer for hypersonic Mach numbers. Therefore, the location of the critical height in the laminar boundary layer may have some effect on the ultimate breakdown of the laminar boundary layer to transitional flow. A knowledge of the location of the critical height would then be of interest in the study of transition. The only method available in the literature, however, for predicting the location of the critical height appears to be the complex and limited stability theory.

It is the purpose of this Note to show that the relatively simple stability parameter proposed by Rouse[10,11] can be used to predict the location of the measured critical layer in a laminar boundary layer. The parameter proposed by Rouse has the following form: $R = (\rho y^2/\mu)\partial u/\partial y$, where all quantities are evaluated locally within the boundary layer. Typical variations of this parameter have been calculated using the local similarity method of Ref. 12 and are shown in Fig. 1 for the conditions noted. From the figure 1, it can be seen that the nondimensional location $[(3^j)^{1/2}y(R_{e,x})^{1/2}/x]$ of the maximum in R depends only on t_w for these particular conditions. Rouse assumed that the point where the parameter reaches a maximum represents the location of maximum instability in the boundary layer or the critical height. Presumably, transition could occur when the parameter reaches a certain critical value. This critical value must be determined from experimental data.

Theoretical values for the critical height are compared with the measured hot wire or hot film data of Refs. 1–7 (taken from Ref. 7) in Fig. 2. From the lower part of the figure, it can be seen that the trend of the critical height with Mach number from the theory is in good agreement with the experimental trend. The upper part of Fig. 2 shows that the absolute level of the data and theory agrees well in the Mach number range from 2 to 8. There is a discrepancy between the theory and data at a Mach number of zero. The reason for this discrepancy is not known, but it may be partly due to the data being obtained in the transition region of the boundary layer.[7] The agreement between the data and theory is fair for Mach numbers above 8. Here again the data in a transitional boundary layer are in poorest agreement with theory.

It should be pointed out that the experimental techniques used to operate the hot wire anemometer could influence the location of the measured critical height for all the data except that of Ref. 1. For example, a constant current anemometer is often operated at a given current setting independent of local fluid properties. This method of operation results in a

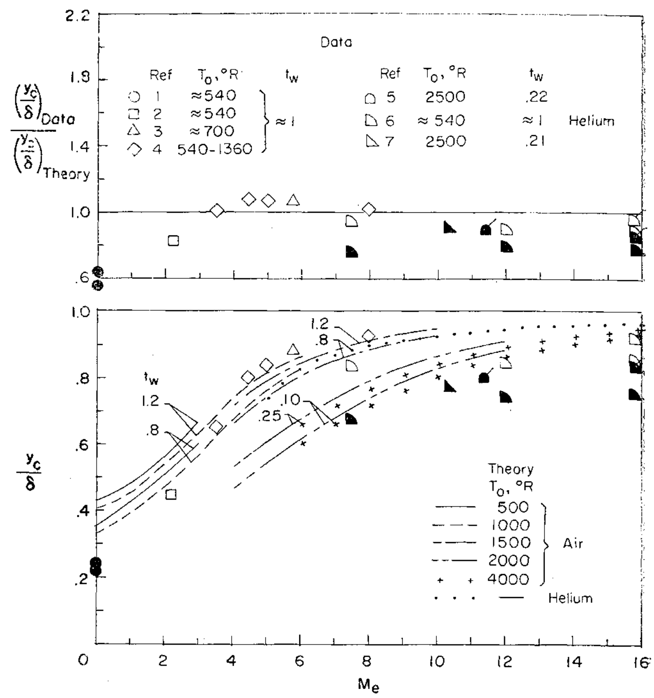

Fig. 2 Location of critical layer in a laminar boundary layer. Flagged symbols represent data taken in a conical nozzle. Filled symbols represent transitional data.

relative decrease in sensitivity of the hot wire in the outer edge of the boundary layer,[3] and this causes the critical heights to be less than they would be if the current settings were properly adjusted across the boundary layer. Nevertheless, the upper portion of Fig. 2 indicates that for $M_e > 3$, the relatively simple stability parameter proposed by Rouse predicts the location of the measured critical height for completely laminar boundary layers to within about 10%.

It is interesting to note that the critical height predicted by the theory increases with increasing Mach number and increasing wall temperature but decreases with increasing total temperature. In fact, the critical height will be a function of any variable which influences the laminar boundary layer profiles including the following quantities: edge Mach number, wall temperature, total temperature or total enthalpy, type of gas, pressure gradient, wall blowing or suction rate, absolute pressure level for real gases, and upstream history of the boundary layer.

Since the present parameter apparently predicts the location of the critical layer in a laminar boundary layer, it may be useful for correlating transition data. To the writer's knowledge this has been attempted in only two boundary-layer transition studies. A modification of this parameter has been used with success to correlate the effect of stream turbulence on transition in Ref. 13. Correlation attempts, using Rouse's original assumptions, were made in Ref. 14 for a wide range of flow conditions, but the results were only fair, possibly because of the wide variety of flow configurations considered. Further attempts to use Rouse's stability parameter to correlate transition data would probably be more successful if data from similar flow configurations were used.

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Real Gas Effects on Shock-Tube Flow Nonuniformity

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An understanding of flow nonuniformity is necessary for meaningful shock-tube testing. One important source of flow nonuniformity is the boundary-layer build-up near the shock-tube wall. The experiments of Duff[1] showed that the shock and contact surfaces approach a limiting separation distance and that the density increases significantly between shock and contact surfaces. This behavior was described analytically by the work of Roshko[2] and Mirels.[3,4] Reference 4 presents calculated results based on an ideal gas for a range of shock Mach numbers. Bertin[5] analyzed boundary-layer induced property variations in a circular shock tube for equilibrium real air operation at shock Mach numbers up to 9.5. Both papers indicate a definite decrease in flow nonuniformity with increase in Mach number.

The present work was undertaken to determine the effect of real gas behavior on flow nonuniformity over a range of Mach numbers. Real gas effects increase with increasing shock Mach number. Physical considerations indicate that the phenomena of dissociation and vibrational excitation would tend to lessen flow nonuniformity. To predict the magnitude of this lessening as a function of shock Mach numbers, the method of Ref. 4 was extended to include real gas effects. Equilibrium flow was considered over a range of Mach numbers. The concept of an equivalent inviscid channel (as described in Ref. 4 and illustrated in Fig. 1) was used for all cases. With the assumption of laminar flow, the channel area can be described by the equation

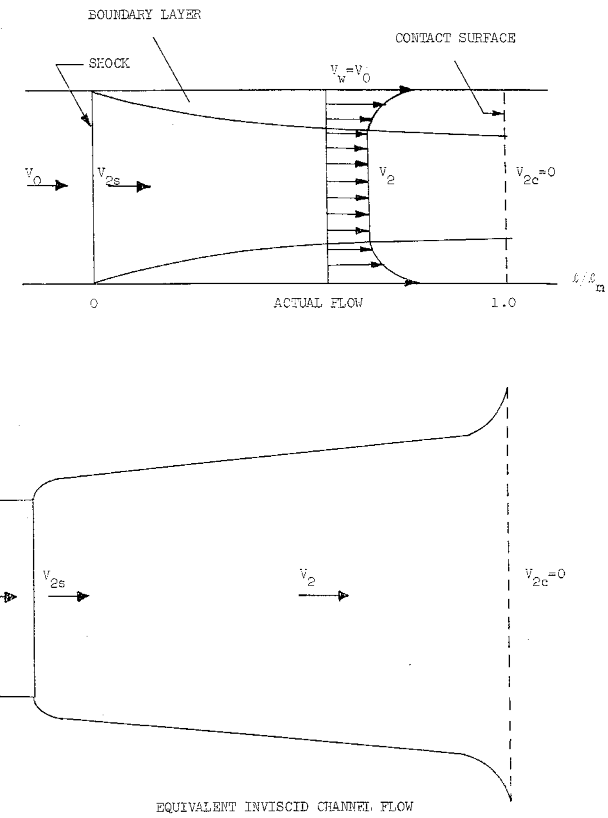

Fig. 1 Actual and equivalent inviscid channel representation of flow in fixed shock coordinates.

$$A_{2s}/A_2 = 1 - (l/l_m)^{1/2} \tag{1}$$

where A_{2s} = cross-sectional area of the channel at the shock front, A_2 = cross-sectional area at distance l behind the shock, l = distance behind the shock, l_m = maximum separation distance between the shock and the contact surface. Calculated results based on such an inviscid channel and presented in Ref. 4 give the variation in flow properties between the shock and contact surface for a laminar boundary layer in an ideal gas. These results are compared to the real gas calculations of the present investigation.

The "ideal dissociating gas" model of Lighthill[6] was used to represent nitrogen in chemical equilibrium for conditions at which dissociation occurs. The accurate range of approximation using this model has a lower limit due to its assumptions regarding vibrational energy and an upper limit imposed by the neglecting of excited electronic states. To obtain reasonably accurate representation using this model, calculations were restricted to shock Mach numbers from 10 to 20.

The assumption of chemical equilibrium is valid provided the length of the relaxation zone λ behind the normal shock is small compared to the maximum separation distance between the shock and the contact surface. Experimental values for the relaxation length in shock-heated nitrogen were found by Allen, Keck, and Camm[7] for initial pressures of 1, 3, and 10 mm of mercury and shock Mach numbers in the range considered in the present work. The values of l_m can be obtained using the results of Ref. 3. Comparison of the values of λ and l_m for an initial pressure of one millimeter of mercury show that for Mach number of 20 in a 4-in.-diam shock tube, the relaxation zone length is less than one percent of l_m. For a Mach number of 15 the relaxation zone occupies

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